A TWO-LEVEL FINITE ELEMENT DISCRETIZATION OF THE STREAMFUNCTION FORMULATION OF THE STATIONARY QUASI-GEOSTROPHIC EQUATIONS OF THE OCEAN

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Abstract. In this paper we propose a two-level finite element discretization of the nonlinear stationary quasi-geostrophic equations, which model the wind driven large scale ocean circulation. Optimal error estimates for the two-level finite element discretization were derived. Numerical experiments for the two-level algorithm with the Argyris finite element were also carried out. The numerical results verified the theoretical error estimates and showed that, for the appropriate scaling between the coarse and fine mesh sizes, the two-level algorithm significantly decreases the computational time of the standard one-level algorithm.

Key words. Quasi-geostrophic Equations, Finite Element Method, Argyris Element, Two-Level algorithms, Streamfunction Formulation.

1. Introduction.

Two-level algorithms are computationally efficient approaches for *finite element* (FE) discretizations of nonlinear partial differential equations [3, 4, 11, 19, 28]. A two-level FE discretization aims to solve a particular nonlinear elliptic equation by first solving the nonlinear system on a coarse mesh and then using the coarse mesh solution to solve the linearized system on a fine mesh. The apeal of such a method is clear; one need only solve the nonlinear equations on a coarse mesh and then use this solution to solve on a fine mesh, thereby reducing computational time without sacrificing solution accuracy. The development of the two-level FE discretization was originally performed by Xu in [28]. Later algorithms were developed for the *Navier-Stokes equations* (NSE) by Layton [19] (see also [11, 12, 26, 20, 29, 21]) and for the Boussinesq equations by Lenferink [22].

As computational power increases complex models are becoming more and more popular for the numerical simulation of oceanic and atmospheric flows. Computational efficiency, however, remains an important consideration for geophysical flows in which long time integration is needed. Thus, simplified mathematical models are central to the numerical simulation of such flows. For example, the *quasi-geostrophic equations* (QGE), a standard mathematical model for wind driven large scale oceanic and atmospheric flows [23, 27], are often used in climate modeling [10].

Most FE discretizations of the QGE are for the streamfunction-vorticity formulation. The reason is that the streamfunction-vorticity formulation allows the use of low order (C^0) FEs, although one needs to discretize two flow variables, the potential vorticity q and the streamfunction ψ. We note that the streamfunction-vorticity formulation is often used in the numerical discretization of the 2D NSE, to which the QGE are similar. Alternatively, one can, instead, use the pure streamfunction formulation of the QGE. The advantage lies in an equation that contains only one flow variable, the streamfunction, ψ, at the price of having to deal with a fourth-order partial differential equation. Thus, its numerical discretization with conforming FEs requires the use of high-order (C^1) FEs, e.g. the Argyris element[1, 6].

The streamfunction formulation of the QGE still suffers from having to solve a large nonlinear system of equations. This is usually done by using a nonlinear solver, such as Newton's method. These nonlinear solvers typically require solving large linear systems multiple times to obtain the solution to the nonlinear system.

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Solving these large linear systems multiple times can be time consuming. Thus, a two-level algorithm greatly reduces computational time over the standard nonlinear solver, since we need only solve the nonlinear system on a coarse mesh and then use that solution to solve a linear system on a fine mesh.

In this paper, we propose a two-level algorithm for the FE discretization of the *streamfunction form of the stationary QGE* (SQGE). This conforming FE discretization is based on the Argyris element. Additionally, we present a rigorous error analysis for the two-level FE discretization. The theoretical error bounds as well as the increased computational efficiency are illustrated numerically for a test problem.

The rest of the paper is organized as follows: In Section 2 we present the SQGE. Then in Section 3 we present the weak formulation of the SQGE, including notation and functional spaces. Next, Section 4 contains the presentation of the one-level FE discretization of the SQGE. In Section 5 we discuss both the two-level algorithm and its application to the SQGE. Next, in Section 6 we provide rigorous error bounds for the two-level FE discretization of the SQGE and we discuss the scaling between the fine mesh, h, and the coarse mesh, H. Then Section 7 includes numerical results which both verify the theoretical error bounds presented in Section 6 and illustrate the computational efficiency of the two-level algorithm over the standard one-level method. Finally, in Section 8 we present our conclusions.

2. Streamfunction Formulation.

The pure streamfunction formulation of the SQGE is

$$Re^{-1}\Delta^2\psi + J(\psi,\Delta\psi) - Ro^{-1}\frac{\partial\psi}{\partial x} = Ro^{-1}F, \quad \text{in } \Omega, \tag{2.1}$$

where

$$\begin{aligned} J(u,v) =& \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}, \\ Ro = \frac{U}{\beta L^2}, &\quad Re^{-1} = \frac{UL}{A} \end{aligned} \tag{2.2}$$

are the Jacobian, Rosby number, Reynolds number, respectively, and β, A, U, and L are the cooeficient in the beta plane approximation, the eddy viscosity, the characteristic velocity scale, and the characteristic length scale, respectively (see [14, 25]).

To completely specify (2.1), we need to impose boundary conditions (see [9, 27, 24] for a careful discussion of this issue). In this report, we consider

$$\psi = \frac{\partial\psi}{\partial\vec{n}} = 0 \quad \text{on } \Omega, \tag{2.3}$$

where $\vec{n}$ represents the outward unit normal to Ω. These are also the boundary conditions used in [17, 11, 13] for the 2D NSE.

3. Weak Formulation.

Now we can derive the weak formulation of the SQGE in streamfunction formulation (2.1). To this end, we first introduce the appropriate functional setting. Let

$$X := H_0^2(\Omega) = \left\{\psi \in H^2(\Omega) : \psi = \frac{\partial\psi}{\partial n} = 0 \text{ on } \partial\Omega\right\}.$$

Multiplying (2.1) by a test function $\chi \in X$ and using the divergence theorem, we get, in the standard way (see [17]), the *weak formulation* of the SQGE in streamfunction

formulation:

$$\begin{aligned} &\textit{Find } \psi \in X \textit{ such that} \\ a(\psi,\chi) + b(\psi;\psi,\chi) &+ c(\psi,\chi) = \ell(\chi), \quad \forall \chi \in X, \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} a(\psi,\chi) &= Re^{-1} \int_\Omega \Delta\psi \Delta\chi \, d\vec{x}, \\ b(\zeta;\psi,\chi) &= \int_\Omega \Delta\zeta \, (\psi_y \chi_x - \psi_x \chi_y) \, d\vec{x}, \\ c(\psi,\chi) &= -Ro^{-1} \int_\Omega \psi_x \chi \, d\vec{x}, \\ \ell(\chi) &= Ro^{-1} \int_\Omega F \, \chi \, d\vec{x}. \end{aligned} \tag{3.2}$$

Lemma 1. *Given ψ, ξ, $\varphi \in H_0^2(\Omega)$ and $F \in H^{-2}(\Omega)$, the linear form ℓ, the bilinear forms a and c, and the trilinear form b are continuous: there exist $\Gamma_1 > 0$ and $\Gamma_2 > 0$ such that*

$$a(\psi,\chi) \leq Re^{-1} |\psi|_2 |\chi|_2 \tag{3.3}$$

$$b(\zeta;\psi,\chi) \leq \Gamma_1 |\zeta|_2 |\psi|_2 |\chi|_2 \tag{3.4}$$

$$c(\psi,\chi) \leq Ro^{-1} \, \Gamma_2 |\psi|_2 |\chi|_2 \tag{3.5}$$

$$\ell(\chi) \leq Ro^{-1} \|F\|_{-2} |\chi|_2. \tag{3.6}$$

For a proof see [7].

For small enough data, one can use the same type of arguments as in [15, 16] to prove that the SQGE in streamfunction formulation (2.1) is well-posed. In what follows we will always assume that the small data condition involving Re, Ro, and F, is satisfied and, thus, that there exists a unique solution ψ to (2.1).

The following stability estimate was proven in [14]:

Lemma 2. *The solution ψ of (2.1) satisifies the following stability estimate:*

$$\|\psi\|_2 \leq Re \, Ro^{-1} \|F\|_{-2}. \tag{3.7}$$

4. Finite Element Formulation. Let $\mathcal{T}^H$ denote a FE triangulation of Ω with meshsize (maximum triangle diameter) H. We consider a *conforming* FE discretization of (3.1), i.e., $X^H \subset X = H_0^2(\Omega)$.

The FE discretization of the SQGE (3.1) reads: Find $\psi^H \in X^H$ such that

$$a(\psi^H,\chi^H) + b(\psi^H;\psi^H,\chi^H) + c(\psi^H,\chi^H) = \ell(\chi^H), \quad \forall \, \chi^H \in X^H. \tag{4.1}$$

Using standard arguments [15, 16], one can prove that, if the small data condition used in proving the well-posedness result for the continuous case holds, then (4.1) has a unique solution ψ^H (see Theorem 2.1 in [7]). Furthermore, one can prove the following stability result for ψ^H using the same arguments as those used in the proof of Lemma 2 for the continuous setting (see [14]).

Lemma 3. *The solution ψ^H of (4.1) satisfies the following stability estimate:*

$$|\psi^H|_2 \leq Re \, Ro^{-1} \, \|F\|_{-2}. \tag{4.2}$$

As noted in Section 6.1 in [8] (see also Section 13.2 in [17], Section 3.1 in [18], and Theorem 5.2 in [5]), in order to develop a conforming FE for the SQGE (3.1), we are faced with the problem of constructing subspaces of $H_0^2(\Omega)$. Since the standard, piecewise polynomial FE spaces are locally regular, this construction amounts in practice to finding FE spaces X^H that satisfy the inclusion $X^H \subset C^1(\overline{\Omega})$, i.e., finding C^1 FEs. Several FEs meet this requirement (see, e.g., Section 6.1 in [8], Section 13.2 in [17], and Section 5 in [5]): the Argyris triangular element, the Bell triangular element, the Hsieh-Clough-Tocher triangular element (a macroelement), and the Bogner-Fox-Schmit rectangular element. In this study we use the Argyris element.

5. Two-Level Algorithm. In this section we propose a two-level FE discretization of the SQGE (3.1). We let X^h, $X^H \subset X = H_0^2(\Omega)$ denote two conforming FE meshes with $H > h$. The two-level algorithm consists of two steps. In the first step, the nonlinear system is solved on a coarse mesh, with mesh size H. In the second step, the nonlinear system is linearized around the approximation found in the first step, and the resulting linear system is solved on the fine mesh, with mesh size h. This procedure is as follows:

Algorithm 1 Two-Level algorithm

Step 1: Solve the following nonlinear system on a coarse mesh for $\psi^H \in X^H$:

$$a(\psi^H, \chi^H) + b(\psi^H; \psi^H, \chi^H) + c(\psi^H, \chi^H) = \ell(\chi^H), \quad \text{for all } \chi^H \in X^H. \tag{5.1}$$

Step 2: Solve the following linear system on a fine mesh for $\psi^h \in X^h$:

$$a(\psi^h, \chi^h) + b(\psi^H; \psi^h, \chi^h) + c(\psi^h, \chi^h) = \ell(\chi^h), \quad \text{for all } \chi^h \in X^h. \tag{5.2}$$

The well-posedness of the nonlinear system was proven in [7], see also [14]. The following lemma proves the well-posedness of the linear system (5.2).

The following theorem provides an error estimate for the approximation in Step 1 of the two-level algorithm (Algorithm 1).

Theorem 1. *Let ψ be the solution of* (3.1) *and ψ^H be the solution of* (5.1). *Furthermore, assume that the following small data condition is satisfied:*

$$Re^{-2}\, Ro \geq \Gamma_1 \|F\|_{-2}.$$

Then the following error estimate holds:

$$|\psi - \psi^H|_2 \leq C(Re, Ro, \Gamma_1, \Gamma_2, F) \inf_{\chi^H \in X^H} |\psi - \chi^H|_2, \tag{5.3}$$

where

$$C(Re, Ro, \Gamma_1, \Gamma_2, F) := \frac{\Gamma_2\, Ro^{-1} + 2Re^{-1} + \Gamma_1\, Re\, Ro^{-1} \|F\|_{-2}}{Re^{-1} - \Gamma_1\, Re\, Ro^{-1} \|F\|_{-2}}.$$

For a proof see [14].

Lemma 4. *Given a solution ψ^H of* (5.1), *the solution, ψ^h, to* (5.2) *exists uniquely.*

Proof. First we introduce the bilinear form $B : X^h \times X^h \to \mathbb{R}$ given by

$$B(\psi^h, \chi^h) = a(\psi^h, \chi^h) + b(\psi^H; \psi^h, \chi^h) + c(\psi^h, \chi^h). \tag{5.4}$$

By Lemma 1

$$B(\psi^h, \chi^h) \leq \left(Re^{-1} + \Gamma_1 |\psi^H|_2 + Ro^{-1}\,\Gamma_2\right) |\psi^h|_2\, |\chi^h|_2, \quad \forall \psi^h, \chi^h \in X^h. \tag{5.5}$$

The stability estimate, for ψ^H, in Lemma 3 and inequality (5.5) imply that B is continuous. Additionally, the fact that $b(\psi^H; \psi^h, \psi^h) = 0$ and $c(\psi^h, \psi^h) = 0$ for all $\psi^h \in X^h$ combined with the Poincaré-Friedrichs inequality gives

$$B(\psi^h, \psi^h) \geq C\, \|\psi^h\|_2, \quad \forall \psi^h \in X^h.$$

Thus, B is coercive. Therefore, by Lax-Milgram, ψ^h exists and is unique. □

In addition to the existence, uniqueness, and stability of the solution to the continuous linear system (Lemma 4) we also have a stability bound for the solution on the discrete fine mesh, h.

Lemma 5. *The solution, ψ^h, to* (5.2) *satisfies the following stability bound:*

$$\|\psi^h\|_2 \leq Re\, Ro^{-1} \|F\|_{-2}. \tag{5.6}$$

Proof. Setting $\chi^h = \psi^h$ in (5.2), and noting that $c(\psi^h, \chi^h) = -c(\chi^h, \psi^h)$, which implies that $c(\psi^h, \psi^h) = 0$, and $b(\psi^H; \psi^h, \psi^h) = 0$ gives

$$\begin{aligned} Re^{-1} \|\psi^h\|_2^2 &= \ell(\psi^h) \Rightarrow \\ \|\psi^h\|_2 &= Re \frac{\ell(\psi^h)}{\|\psi^h\|_2} \leq Re Ro^{-1} \|F\|_{-2}, \end{aligned}$$

where in the last inequality we used (3.6). Therefore, it follows that

$$\|\psi^h\|_2 \leq Re\, Ro^{-1} \|F\|_{-2}.$$ □

6. Error Bounds. The main goal of this section is to develop a rigorous numerical analysis for the two-level algorithm (Algorithm 1). The proof for the error bounds follows the pattern presented in [11].

We first introduce an improved bound on the trilinear form $b(\zeta; \xi, \chi)$. To this end, we use the following discrete Sobolev inequality [11]:

$$\|\nabla \varphi^h\|_{L^\infty} \leq c \sqrt{|\ln(h)|}\, |\varphi^h|_2 \quad \forall \varphi^h \in X^h. \tag{6.1}$$

The following lemma will be useful in determining the error bounds for Step 2 of the two-level algorithm. The first lemma which corresponds to Lemma 5.1 in [11], follows from (6.1) and (3.4) and places error bounds on the trilinear form $b(\psi; \chi^h, \xi)$:

Lemma 6. *For any $\chi^h \in X^h$, the following inequality holds:*

$$|b_0(\psi; \chi^h, \xi)| \leq C \sqrt{|\ln(h)|}\, |\psi|_2 |\xi|_1 |\chi^h|_2,$$

where

$$b_0(\xi; \chi, \psi) = \int_\Omega (\chi_y \xi_{xy} - \xi_x \chi_{yy}) \psi_y - (\xi_y \chi_{yx} - \xi_y \chi_{xx}) \psi_x \, d\vec{x}. \tag{6.2}$$

The following lemma, which corresponds to Lemma 5.6 in [11], will be useful for proving the error bounds for Algorithm 1, by allowing one to permute the terms of the trilinear term:

Lemma 7. *For ψ, ξ, $\chi \in H_0^2(\Omega)$, we have*

$$b(\psi; \xi, \chi) = b_0(\xi; \chi, \psi) - b_0(\chi; \xi, \psi). \tag{6.3}$$

The following theorem gives the error bound after Step 2 of the two-level algorithm (Algorithm 1) and is the main result of this paper. The proof of this theorem is similar to the proof of Theorem 5.2 in [11].

Theorem 2. *Let ψ be the solution to* (3.1) *and ψ^h the solution to* (5.2)*. Then ψ^h satisfies*

$$|\psi - \psi^h|_2 \le C_1 \inf_{\lambda^h \in X^h} |\psi - \lambda^h|_2 + C_2 \sqrt{|\ln h|}\, |\psi - \psi^H|_1, \tag{6.4}$$

where $C_1 = 2 + Re\, Ro^{-1}\, \Gamma_2 + Re^2 Ro^{-1} \Gamma_1 \|F\|_{-2}$ and $C_2 = 2Re^2 Ro^{-1} \Gamma_1 C \, \|F\|_{-2}$.

Proof. Subtracting (5.2) from (3.1) and letting $\chi = \chi^h \in X^h \subset X$ yields the error equation:

$$a(\psi - \psi^h, \chi^h) + b(\psi; \psi, \chi^h) - b(\psi^H; \psi^h, \chi^h) + c(\psi - \psi^h, \chi^h) = 0, \quad \forall \chi^h \in X^h. \tag{6.5}$$

Now, adding the terms

$$-b(\psi; \psi^h, \chi^h) + b(\psi; \psi^h, \chi^h)$$

to (6.5) gives

$$a(\psi - \psi^h, \chi^h) + b(\psi; \psi - \psi^h, \chi^h) + b(\psi - \psi^H; \psi^h, \chi^h) + c(\psi - \psi^h, \chi^h) = 0, \quad \forall \chi^h \in X^h. \tag{6.6}$$

Take $\lambda^h \in X^h$ arbitrary and define $e := \psi - \psi^h = \eta - \Phi^h$, where $\Phi^h = \psi^h - \lambda^h$ and $\eta = \psi - \lambda^h$. Equation (6.6) becomes

$$\begin{aligned} a(\Phi^h, \chi^h) + b(\psi; \Phi^h, \chi^h) + c(\Phi^h, \chi^h) \quad &= \\ a(\eta, \chi^h) + b(\psi; \eta, \chi^h) + b(\psi - \psi^H; \psi^h, \chi^h) + c(\eta, \chi^h), &\quad \forall \chi^h \in X^h. \end{aligned} \tag{6.7}$$

Since (6.7) holds for any $\chi^h \in X^h$ it holds in particular for $\chi^h = \Phi^h \in X^h$, which implies

$$\begin{aligned} &a(\Phi^h, \Phi^h) + b(\psi; \Phi^h, \Phi^h) + c(\Phi^h, \Phi^h) = \\ &\quad a(\eta, \Phi^h) + b(\psi; \eta, \Phi^h) + b(\psi - \psi^H; \psi^h, \Phi^h) + c(\eta, \Phi^h). \end{aligned} \tag{6.8}$$

Note that $c(\psi, \chi) = -c(\chi, \psi)$ which implies $c(\Phi^h, \Phi^h) = 0$. Also, $b(\psi; \chi, \chi) = 0$ and so (6.8) becomes

$$a(\Phi^h, \Phi^h) = a(\eta, \Phi^h) + b(\psi; \eta, \Phi^h) + b(\psi - \psi^h; \psi^h, \Phi^h) + c(\eta, \Phi^h). \tag{6.9}$$

Now rewriting $b(\psi - \psi^h; \psi^h, \Phi^h)$ using Lemma 7 yields

$$\begin{aligned} a(\Phi^h, \Phi^h) = {} & a(\eta, \Phi^h) + b(\psi; \eta, \Phi^h) \\ & + b_0(\psi^h; \Phi^h, \psi - \psi^h) + b_0(\Phi^h; \psi^h, \psi^H - \psi) + c(\eta, \Phi^h). \end{aligned} \tag{6.10}$$

Using the error bounds given in Lemmas 1, 2, 4, and 6 in (6.10) gives

$$\begin{aligned} Re^{-1} |\Phi^h|_2^2 \le {} & Re^{-1} |\eta|_2 \, |\Phi^h|_2 + \Gamma_1 \, |\psi|_2 \, |\eta|_2 \, |\Phi^h|_2 \\ & + 2\Gamma_1 \, C \, |\psi^h|_2 \, |\Phi^h|_2 \, |\psi - \psi^H|_1 \sqrt{|\ln(h)|} + Ro^{-1} \, \Gamma_2 |\eta|_2 \, |\Phi^h|_2 \\ = {} & \left(Ro^{-1} \, \Gamma_2 + Re^{-1} + \Gamma_1 \, |\psi|_2\right) |\eta|_2 \, |\Phi^h|_2 \\ & + 2\Gamma_1 \, C \, |\psi^h|_2 \, |\Phi^h|_2 |\psi - \psi^H|_1 \sqrt{|\ln(h)|}. \end{aligned}$$

Simplifying by $|\phi^h|_2$ and using the stability estimates (3.7) in Lemma 2 and (4.2) in Lemma 3 gives

$$\begin{aligned} |\Phi^h|_2 \le & \left(1 + Re\, Ro^{-1}\, \Gamma_2 + Re^2 Ro^{-1} \Gamma_1 \, \|F\|_{-2}\right) |\eta|_2 \\ & + 2Re^2 Ro^{-1} \Gamma_1\, C\, \|F\|_{-2}\, |\psi - \psi^H|_1 \sqrt{|\ln(h)|}. \end{aligned} \tag{6.11}$$

Adding $|\eta|_2$ to both sides of (6.11) and using the triangle inequality gives

$$\begin{aligned} |\psi - \psi^h|_2 \le & \left(2 + Re\, Ro^{-1}\, \Gamma_2 + Re^2 Ro^{-1} \Gamma_1 \, \|F\|_{-2}\right) |\eta|_2 \\ & + 2Re^2 Ro^{-1} \Gamma_1\, C\, \|F\|_{-2}\, |\psi - \psi^H|_1 \sqrt{|\ln(h)|}. \end{aligned} \tag{6.12}$$

Taking the infimum over $\lambda^h \in X^h$ in (6.12) yields the estimate (6.4). $\square$

In what follows, we consider both X^h and X^H Argyris FE spaces. We emphasize, however, that both Algorithm 1 and the error estimate in Theorem 2 remain valid for other conforming FE spaces, e.g. the Bell element, the Hsieh-Clough-Tocher element, or the Bogner-Fox-Schmit element.

For the Argyris triangle; we have the following inequalities, which follow from approximation theory [2] and Theorem 6.1.1 in [8]:

$$\begin{aligned} |\psi - \psi^h|_j &\le C h^{6-j}, \\ |\psi - \psi^H|_j &\le C H^{6-j}, \end{aligned} \tag{6.13}$$

where $j = 0, 1, 2$ and ψ, the solution of (3.1), is assumed to satisfy $\psi \in H^6(\Omega) \cap H_0^2(\Omega)$.

Corollary 1. *Let X^h, $X^H \subset H_0^2(\Omega)$ be Argyris finite elements. Then ψ^h, the solution of the two-level algorithm (Algorithm 1) satisfies the following error estimate:*

$$|\psi - \psi^h|_2 \le C_1 h^4 + C_2 \sqrt{|\ln(h)|}\, H^5. \tag{6.14}$$

Proof. This follows directly by substituting the inequalities (6.13) into (6.4). $\square$

7. Numerical Results. The goal of this section is two-fold: first, we illustrate the computational efficiency of the two-level method, and second, we verify the theoretical rates of convergence developed in Section 6. To illustrate the computational efficiency of the two-level method, we compare solution times for the full nonlinear one-level problem and for the two-level method applied to the SQGE. We choose coarse mesh/fine mesh pairs such that the ratio is 1/2. To verify the theoretical rates of convergence, we compare the theoretical error estimates to the observed rates of convergence from our numerical tests. For the one-level problem we rely on our original code that was benchmarked in [14].

To this end we apply the two-level method to the SQGE (2.1) with $Re = Ro = 1$ and exact solution

$$\psi(x, y) = (\sin 4\pi x \cdot \sin 2\pi y)^2 . \tag{7.1}$$

Additionally, the homogeneous boundary conditions are $\psi = \dfrac{\partial \psi}{\partial \vec{n}} = 0$ and the forcing function F corresponds to the exact solution (7.1). These boundary conditions and exact solution will be used in all the two-level tests that follow.

7.1. Practical Consideration. A key part of two-level algorithms is accessing a previous coarse mesh solution, i.e. finding the parent element given a child element. This step can negate any performance benefits if not implemented wisely. Indeed, let n be the number of elements in the FE discretization. For the unit square, a naïve search across every element takes $O(n/2)$ operations. This procedure may be improved with a binary search, which is summarized in Algorithm 2.

We note that every element on the fine mesh corresponds to exactly one element on the coarse mesh. However, a coarse mesh element may correspond to multiple elements on the fine mesh.

Algorithm 2 Given an element on the fine mesh determine the parent element on the coarse mesh.

Before examining the fine mesh, sort the coarse mesh elements by their centroid values.

Step 1: Select an element on the fine mesh and compute its centroid.

Step 2: Perform a binary search across the coarse mesh elements until the difference between the x-values of the fine mesh centroid and coarse mesh centroids is less than H, the coarse mesh step size. There should be many elements that fit this condition; save them as a list.

Step 3: Search through this list until we find the correct coarse mesh element (that is, the centroid of the fine-mesh element is an interior point of the correct coarse mesh element).

For the considered unit square, the binary search will examine on average $\log(n)$ elements, while the linear search component involves at most $\sqrt{n}/2$ elements. Therefore the search requires a $O(\sqrt{n}/2)$ number of element checks. Profiling results indicate that using Algorithm 2 to identify parent elements takes much less time than either setting up or solving the systems, so this approach is fast enough that lookup time does not contribute significantly to overall solution time.

7.2. Computational Efficiency. To illustrate the computational efficiency of the two-level method, we compare the simulation time for the standard one-level method (i.e. the full nonlinear system, without the two-level method) with the simulation time for the two-level method.

In Table 7.1, the L^2-norm of the error (e_{L^2}), the H^2-norm of the error (e_{H^2}) and the simulation times are listed for various mesh sizes. For each fine mesh, we choose a coarse mesh that ensures the same order of magnitude for the errors in the one-level and two-level methods. For small values of the fine mesh size, h, the two-level method was significantly faster than the one-level method. The errors in the H^2-norm were nearly identical, while the error in the L^2-norm were generally of the same order of magnitude. We also note that the tolerance in Newton's method seems to cause a plateau in the L^2-norm of the error. The results in Table 7.1 are illustrated graphically in Figure 7.1. In this figure the simulation times of the one-level method (green) and of the two-level method (blue) are displayed for all the pairs (h, H) in Table 7.1. Figure 7.1 clearly shows that as the number of degrees of freedom (DoFs) increases, the computational efficiency of the two-level method increases as well.

7.3. Rates of Convergence. The goal of this subsection is to verify, numerically, the theoretical rates of convergence in (6.14) of Corollary 1. Unlike the theoretical error estimates for the one-level method developed in [14], for the two-level method we must verify rates of convergence for two different meshes: the fine mesh, h, and the coarse mesh, H.

H	h	DoFs, H	DoFs, h	e_{L^2}	e_{H^2}	time, s
–	0.05146	–	4362	4.286×10^{-8}	1.648×10^{-3}	3.328
0.1083	0.05146	1158	4362	1.092×10^{-7}	1.709×10^{-3}	2.372
–	0.02561	–	16926	5.748×10^{-10}	1.009×10^{-4}	19.92
0.05146	0.02561	4362	16926	7.691×10^{-10}	1.016×10^{-4}	11.82
–	0.01597	–	43074	4.751×10^{-11}	1.793×10^{-5}	55.69
0.03384	0.01597	10983	43074	5.267×10^{-11}	1.797×10^{-5}	33.19
–	0.01277	–	66678	8.66×10^{-12}	6.207×10^{-6}	102.4
0.02561	0.01277	16926	66678	9.686×10^{-12}	6.217×10^{-6}	59.03
–	0.009659	–	116614	3.876×10^{-12}	2.382×10^{-6}	161.7
0.02035	0.009659	29501	116614	6.836×10^{-12}	2.385×10^{-6}	95.93
–	0.007959	–	170598	4.791×10^{-12}	1.111×10^{-6}	325.1
0.01597	0.007959	43074	170598	9.087×10^{-12}	1.112×10^{-6}	172.3
–	0.006854	–	230574	1.79×10^{-11}	6.16×10^{-7}	401.7
0.01436	0.006854	58131	230574	1.3×10^{-11}	6.163×10^{-7}	219.5
–	0.006374	–	264678	3.912×10^{-11}	3.846×10^{-7}	559.7
0.01277	0.006374	66678	264678	2.309×10^{-11}	3.848×10^{-7}	291.9
–	0.005264	–	389994	3.85×10^{-11}	2.086×10^{-7}	753.4
0.01101	0.005264	98133	389994	6.495×10^{-11}	2.087×10^{-7}	397.7

TABLE 7.1

Comparison of one-level and two-level methods: the L^2-norm of the error (e_{L^2}), the H^2-norm of the error (e_{H^2}) and simulation times.

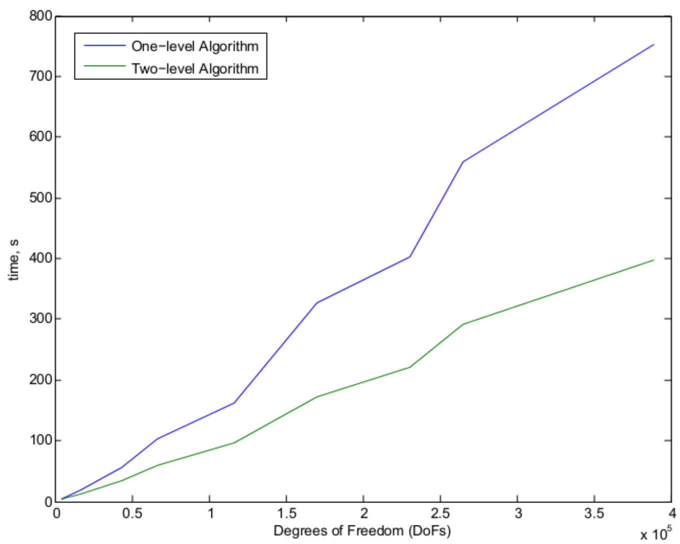

FIG. 7.1. *The simulation times of the one-level method (green) and of the two-level method (blue) are displayed for all the pairs (h, H) in Table 7.1.*

To verify, numerically, the theoretical rate of convergence with respect to H, given in (6.14), we fix h to a small value and we vary H. Thus, the total error in (6.14) will be dominated by the H term, i.e. the total error will of order $O(H^5)$. In Table 7.2, we fix $h = 0.0063$ and we vary H. The error in the L^2-norm (e_{L^2}), the error in the H^2-norm (e_{H^2}), and the rate of convergence with respect to H are listed in Table 7.2.

H	h	DoFs, H	DoFs, h	e_{L^2}	e_{H^2}	H^2 order
0.43	0.0063	350	296710	2.26×10^{-3}	4.54×10^{-1}	–
0.21	0.0063	1270	296710	4.39×10^{-5}	1.755^{-2}	4.45
0.10	0.0063	4838	296710	1.86×10^{-7}	4.92×10^{-4}	5.04
0.05	0.0063	18886	296710	2.11×10^{-9}	1.32×10^{-5}	5.2
0.025	0.0063	74630	296710	$2..18 \times 10^{-9}$	6.02×10^{-7}	4.45

TABLE 7.2

Two-level method: the L^2-norm of the error (e_{L^2}), the H^2-norm of the error (e_{H^2}), and the convergence rate with respect to H

The rate of convergence follows the theoretical rate predicted in (6.14) (i.e. fifth-order). For the last mesh pair, however, the rate of convergence appears to drop off. This occurs because, for small values of H, the total error in (6.14) is not dominated anymore by the H term.

To verify, numerically, the theoretical rate of convergence with respect to h, given in (6.14), we must proceed with caution. The reason is that a straightforward approach would fix H and let h go to zero. This approach, however, would fail, since the H term would eventually dominate the total error. To avoid this, we consider the following scaling between the mesh sizes:

$$H = C\,h, \tag{7.2}$$

where $C > 1$. The scaling in (7.2) implies that the total error in (6.14) is of order $O(h^4)$. Indeed, the second term on the right hand side of (6.14) now scales as follows:

$$\begin{aligned} C_2\sqrt{|\ln(h)|}\,H^5 &\approx C_2 C\sqrt{|\ln(h)|}\,h^5 \\ &\approx O(h^4), \end{aligned} \tag{7.3}$$

where in the last relation in (7.3) we used the fact that $\sqrt{|\ln(h)|}\,h \to 0$ when $h \to 0$ (which follows from l'Hospital's rule).

Remark 1. *We emphasize that the scaling in* (7.2) *is not needed in the two-level algorithm. We only use it in this subsection to monitor the convergence rate with respect to h.*

In this subsection, we consider $C = 2$ in (7.2). We note, however, that any other constant $C > 1$ could be used in (7.2). With this choice, we are now ready to verify, numerically, the theoretical rate of convergence with respect to h given in (6.14), which, as shown in (7.3), will be of order $O(h^4)$. In Table 7.3, for various mesh size pairs ($H = 2h, h$), we list the L^2-norm of the error (e_{L^2}), the H^2-norm of the error (e_{H^2}), and the rate of convergence. The rate of convergence follows the theoretical rate predicted in (6.14) (i.e. fourth-order).

8. Conclusions. In this paper, we proposed a two-level FE discretization of the (nonlinear) stationary quasi-geostrophic equations. The two-level algorithm consists of two steps. In the first step, the nonlinear system is solved on a coarse mesh. In the second step, the nonlinear system is linearized around the approximation found in the first step, and the resulting linear system is solved on the fine mesh.

Rigorous error estimates for the two-level FE discretization were derived. These estimates are optimal in the following sense: for an appropriately chosen scaling between the coarse mesh, H, and the fine mesh, h, the error in the two-level method

H	h	DoFs, H	DoFs, h	e_{L^2}	e_{H^2}	H^2 order
1.1	0.43	38	106	1.31×10^{-2}	5.832×10^{0}	–
0.43	0.21	106	350	2.25×10^{-3}	6.61×10^{-1}	2.56
0.21	0.10	350	1270	4.40×10^{-5}	3.65×10^{-2}	3.96
0.10	0.05	1270	4838	1.90×10^{-7}	2.00×10^{-3}	4.10
0.050	0.025	4838	18886	8.95×10^{-10}	1.20×10^{-4}	4.04
0.025	0.013	18886	74630	1.36×10^{-10}	7.40×10^{-6}	4.02
0.013	0.0063	74630	296710	2.22×10^{-9}	4.68×10^{-7}	3.99

TABLE 7.3

Two-level method: the L^2-norm of the error (e_{L^2}), the H^2-norm of the error (e_{H^2}), and the convergence rate with respect to h

is of the same order as the error in the standard one-level method (i.e. solving the nonlinear system directly on the fine mesh).

Numerical experiments for the two-level algorithm, with the Argyris element, were also carried out. The numerical results verified, numerically, the theoretical error estimates, both with respect to the coarse mesh size, H, and the fine mesh size, h. Furthermore, the numerical results showed that, for an appropriate scaling between the coarse and fine meshes, the two-level method significantly decreases the computational time of the standard one-level method.

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